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An Explicit Solution to the Powered Flight Dynamics of a Rocket Vehicle

31 OCTOBER 1962

Prepared by DUNCAN MAC PHERSON

Prepared for COMMANDER SPACE SYSTEMS DIVISION
UNITED STATES AIR FORCE
Inglewood, California

ENGINEERING DIVISION • AEROSPACE CORPORATION
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AN EXPLICIT SOLUTION TO THE POWERED FLIGHT
DYNAMICS OF A ROCKET VEHICLE

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ABSTRACT

This document presents a general explicit solution to the powered flight dynamics of a rocket vehicle having constant thrust and constant effective exhaust velocity, and demonstrates the use of this solution in guidance techniques.

CONTENTS

1. INTRODUCTION

In the latter part of 1959, the author developed an explicit solution to the powered flight dynamics of a rocket vehicle having constant thrust and constant effective exhaust velocity. This solution was used as the basis for the guidance equations for the Mercury/Atlas flights. Since that time, guidance equations based on this solution have been developed for use on Project Gemini and Ranger/Mariner, as well as for other programs of a more classified (security) nature.

This explicit solution and its application to guidance equations was documented in Reference (1).* Since that time, many people have examined this document as well as other techniques for guiding a continuously burning vehicle to prescribed values of altitude and velocity at burnout. As a result of these investigations it appears clear that for the class of vehicles under consideration:

(1) any guidance technique which accurately controls burnout altitude and velocity vector must be based on an explicit solution of the powered flight dynamics.

(2) the solution which was originally developed and documented in Reference (1) is essentially definitive (the details of mechanization will of course depend to a certain extent on particular mission requirements).

Although the notation in this document differs from that used in Reference (1) there are no non-trivial changes in the subject matter. Some additional information or explanatory material has been provided for various peripheral areas because these omissions have resulted in considerable confusion by many

*"An Explicit Method of Guiding a Vehicle from an Arbitrary Initial Position and Velocity to a Prescribed Orbit," by D. MacPherson, dated 13 February 1961, Report No. TDR-594(1565-01)TN-1.

readers of Reference (1). The order of presentation in the Basic Vehicle Kinematics section has been changed for greater clarity (the manner of presentation used here was suggested by Mr. M. Tangora). Various small errors in Reference (1) were discovered by a number of people; these have hopefully been corrected. Mr. W. Brocato has been especially helpful in this regard.

This explicit solution to the powered flight dynamics of a rocket vehicle is of fundamental importance for guidance equations on sophisticated missions; because the total effect of control commands is to determine the position and velocity of the vehicle at that point where ability to control terminates (it is obvious that only certain restricted values of position and velocity can be obtained). The values of position and velocity that are desired or required at control termination (thrust termination) may be specified in a variety of ways. On an ICBM mission, for example, it is possible to make the velocity at thrust termination a function of position at thrust termination, and thereby eliminate the need for position control. On the other hand, burnout position and velocity are independently specified on many space missions. The explicit solution contained herein can be utilized to attain these desired conditions at thrust termination regardless of the origin of the specification (the specification may be considered another part of the guidance equations).

In the development it is assumed that vehicle thrust attitude changes (or control) take place in two mutually perpendicular planes which are conventionally designated as pitch and yaw. It is assumed that the thrust magnitude cannot be controlled (except by termination).

This document has been written in as general a manner as possible with emphasis placed on the principles involved rather than on the sophistications and details of mechanization which might be desirable when using this solution in a detailed set of guidance equations. For example, the time difference between generated control commands and effective time of data and/or effective time of command enactment should be accounted for; but this has not been discussed because a bookkeeping technique of this type is not conceptually difficult or fundamental to the system.

2. BASIC VEHICLE KINEMATICS

Newton's law applied to rocket thrust gives:

$$F_T = m a_T = -c^* \frac{dm}{dt} \tag{2.1}$$

The vehicle thrust and effective exhaust velocity will be assumed constant in the following analysis. Although these assumptions usually approximate reality quite satisfactorily, restrictive assumptions of this type are not conceptually necessary; it is, however, necessary (for practical purposes) to have the terms in Equation (2.1) in sufficiently simple form for the necessary analytical treatment. With the above assumption

$$m_1 = m - \frac{F_T}{c^*} (t_1 - t) = m - \frac{m_1 a_{T1}}{c^*} (t_1 - t)$$

where t is present time and t_1 is some future time.

Then

$$m = m_1 \left[1 + \frac{a_{T1}}{c^*} (t_1 - t) \right] \tag{2.2}$$

so

$$a_T = \frac{a_{T1}}{1 + \frac{a_{T1}}{c^*} (t_1 - t)} \tag{2.3}$$

The equivalent relationship

$$a_{T1} = \frac{a_T}{1 - \frac{a_T}{c*}(t_1 - t)} \tag{2.4}$$

will be more convenient at times.

The following dimensionless variable is of fundamental importance.

$$U \equiv - \frac{a_{T1}}{c*}(t_1 - t) \tag{2.5}$$

Note that

$$\frac{dU}{dt} \equiv \dot{U} = \frac{a_{T1}}{c*} \tag{2.6}$$

The constant value of $\dot{U}$ is in a sense a measure of the vehicle's dynamic characteristics. Then "time to go" until $t = t_1$ can be found from

$$T_U \equiv t_1 - t = - \frac{U}{\dot{U}} \tag{2.7}$$

The evaluation of T_U will be discussed after the immediately following development of an appropriate expression for use in the evaluation of U.

The acceleration when U = 0 is a_{T1} and will be redefined as a_U. Then from Equations (2.3), (2.5), and (2.6)

$$a_U = \dot{U}c* \tag{2.8}$$

$$a_T = \frac{a_U}{1 - U} \tag{2.9}$$

The general equation of motion along $\vec{V}$ will now be developed.

The general vehicle force diagram is shown in Figure 2.1.

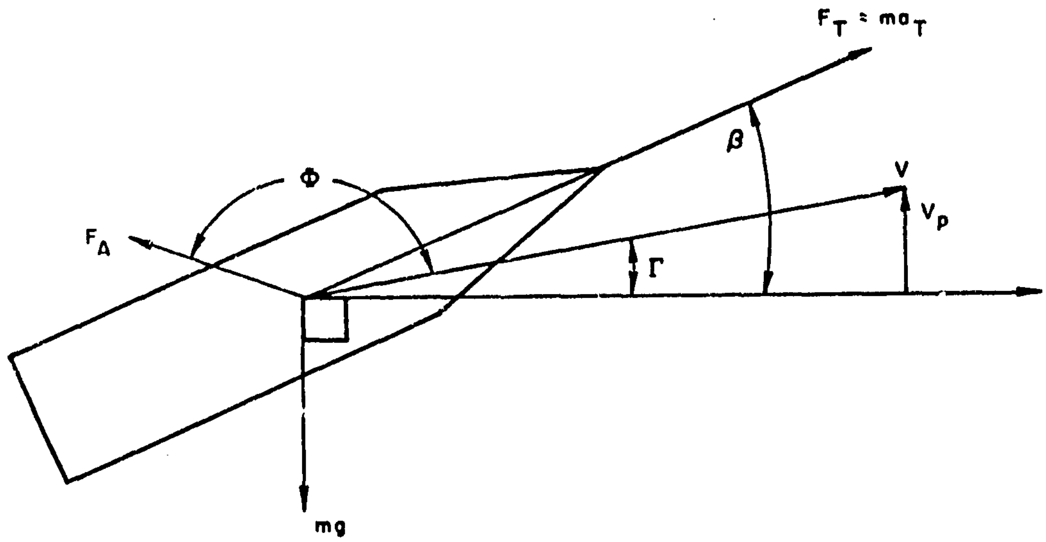

Figure 2.1

These forces will be divided into two components; one parallel to, and one perpendicular to velocity. The latter component is of no interest in the immediate discussion since it changes only the direction and not the magnitude of the velocity. The force balance along $\vec{V}$ may be written as:

$$\frac{dV}{dt} = a_T \cos(\beta - \Gamma) - g \sin\Gamma + \frac{F_A}{m} \cos\Phi$$

Defining

$$a_L \equiv a_T - \frac{dV}{dt} = a_T \left[1 - \cos(\beta - \Gamma)\right] + g \sin\Gamma - \frac{F_A}{m} \cos\Phi \tag{2.10}$$

Then

$$a_T = \frac{dV}{dt} + a_L$$

and

$$\int_t^{t_1} a_T dt = \int_t^{t_1} \frac{dV}{dt}\, dt + \int_t^{t_1} a_L\, dt = V_1 - V + \int_t^{t_1} a_L\, dt \qquad (2.11)$$

The value of a_L may be considered physically as the acceleration "lost" due to angle of attack, drag, and gravity.

From Equations (2.8) and (2.9)

$$\int_t^{t_1} a_T dt = c^* \ln (1 - U) \qquad (2.12)$$

combining Equation (2.11) and (2.12) (where V_f is desired final velocity):

$$U = 1 - e^{\left[-\frac{V_1 - V + \int_t^{t_1} a_L dt}{c^*}\right]}$$

Although the above equation is valid it is not useful because the integral is not known exactly. This is unsatisfactory not only for its own sake but also because it masks the relationship between V_1 and V_f. However, these difficulties can be overcome by the following artifice:

Define

$$V'_L \equiv V_1 - V_f + \int_t^{t_1} a_L dt$$

Then

$$U = 1 - e^{\frac{V_f \cdot \quad + V_L'}{c^*}}$$

The magnitude of V_L' is still unknown. However, the derivative at any time is simply

$$\frac{dV_L'}{dt} = -a_L$$

or numerically

$$V_{L_n}' \approx V_{L_{n-1}}' - a_{L_n} \tau$$

If V_L' were known initially, the above equation could be used to find proper succeeding values. This technique will be used as defined by the following equations:

$$U = 1 - e^{\frac{V_f - V + V_L}{c^*}} \tag{2.13}$$

where

$$V_{L_n} = V_{L_{n-1}} - a_L \tau \tag{2.14}$$

a_L is from Equation (2.10)

τ is length of computation cycle

V_{L_o} is a preselected constant

The value of V_{L_0} may be arbitrarily chosen; however, it is desirable to use a reasonably realistic value so that V_L is near zero when $V = V_f$. It is obvious from Equation (2.13) that when $V = V_f$

$$U_f = 1 - e^{\frac{V_{Lf}}{c^*}}$$

and that elapsed time until $V = V_f$ is

$$T_E = \frac{U_f - U}{\dot{U}}$$

The uncertainty in V_{Lf} is reflected in U_f in the above mechanization. Although the above mechanization is valid, the following alternative has generally been used in the past.

The time difference δT_U between the points where $V = V_1$ and $V = V_f$ is

$$\delta T_U = \frac{-V_{Lf}}{\bar{a}_f} \tag{2.15}$$

where $\bar{a}_f$ is the average thrust acceleration between times t_1 and t_f. Using Equation (2.9)

$$\bar{a}_f = \frac{\int_{t_1}^{t_1 + \delta T_U} a_T\, dt}{\delta T_U} = \frac{a_U}{\delta T_U} \int_{t_1}^{t_1 + \delta T_U} \frac{dt}{1 - U} = a_U \frac{\ln (1 - \dot{U} \delta T_U)}{-\dot{U} \delta T_U} \tag{2.16}$$

Then the time interval until velocity equals desired velocity is (see Figure 2.2)

$$T_E = T_U + \delta T_U \tag{2.17}$$

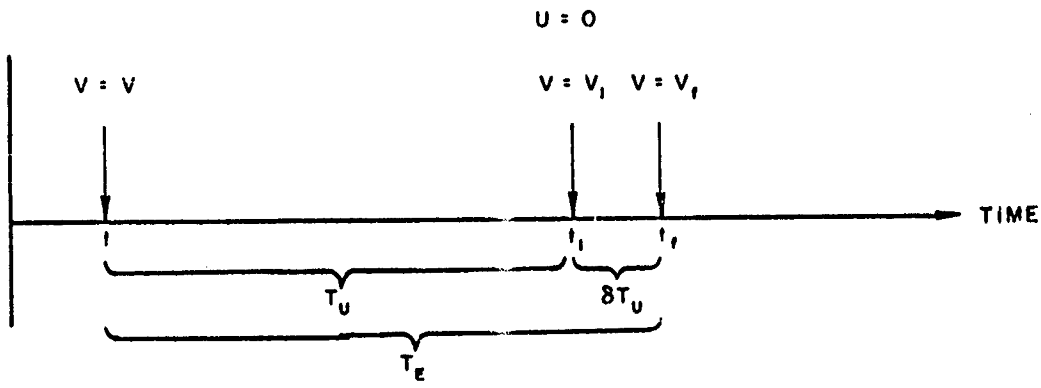

Figure 2.2

The time interval T_U can be mechanized as (from Equation 2.7)

$$T_U = -\frac{U}{\dot{U}}$$

Since $\dot{U}$ can be found by differencing successive values of U, T_U can be determined from knowledge of c* and measurements of velocity only. (See Appendix A for effect of error in knowledge of c*.) It should especially be noted that while the thrust must be constant, it need not be known. Since U is linear in time (from Equation (2.5)), it is an appropriate quantity on which to apply smoothing, and smoothing should be applied on U in preference to an equivalent nonlinear quantity (V for example).

For any mechanization it is necessary to evalua V_{Lf}. Since it is clear that

$$V_{Lf} = V_L - \int_t^{t_f} a_L dt \tag{2.18}$$

This integral must be predicted.

It is obvious that the magnitude of V_{Lf} will not be known exactly when T_E is large; but as T_E becomes smaller the value of the integrand in Equation (2.18) not only becomes more predictable but also requires prediction for shorter periods of time. Techniques which have been used in this prediction are discussed in Appendix D.

The characteristics of T_U and δT_U which have been cited lead to the conclusion that the percentage error in T_E is always small. Simulation has verified this conclusion by consistently demonstrating that errors in T_E are less than (and usually considerably less than) one per cent in the absence of errors in position and velocity (see simulation results in Section 4). It should be noted that the separation of the thrust effects and the "loss" effects (into T_U and δT_U or the equivalent) is necessary in a practical mechanization in order to achieve stability. This separation provides a decoupling which causes first order perturbations in loss estimates to have only second order effects on time to go (T_E) estimates. If decoupling is not provided, perturbations in loss estimates corrupt the estimate of thrust acceleration which causes large variations in time to go. A combination of small loss perturbations and heavy smoothing on time to go can prevent instability, but this is not a desirable mechanization for obvious reasons.

Note that Equation (2.9) gives thrust acceleration in terms of the same smoothed quantities that determine T_U, and that this thrust acceleration can be extrapolated

backwards or forwards in time with no more error (percentagewise) than exists in the current estimate. In particular, the acceleration at engine cutoff is

$$a_f = \frac{a_U}{1 - \dot{U}\delta T_U} \tag{2.19}$$

If thrust and structural mass of the vehicle are known, Equation (2.19) can be used to predict the mass of propellant remaining at engine cutoff.

Figure 2.3 is a collection of equations from this section which are needed for computation of time-to-go and steering commands. It is not suggested that this is the exact set of equations which would be mechanized. Note that since $\dot{U}$ is used in the solution as well as U, the mechanization of Eq. (2.13) should be a truncation of the exponential series instead of a curve fit.

It will be convenient to note for future reference that the total integral of thrust acceleration between times t and t_f is

$$\Delta V_E = \int_t^{t_f} a_T dt = V_f - V + V_L - V_{Lf} \tag{2.20}$$

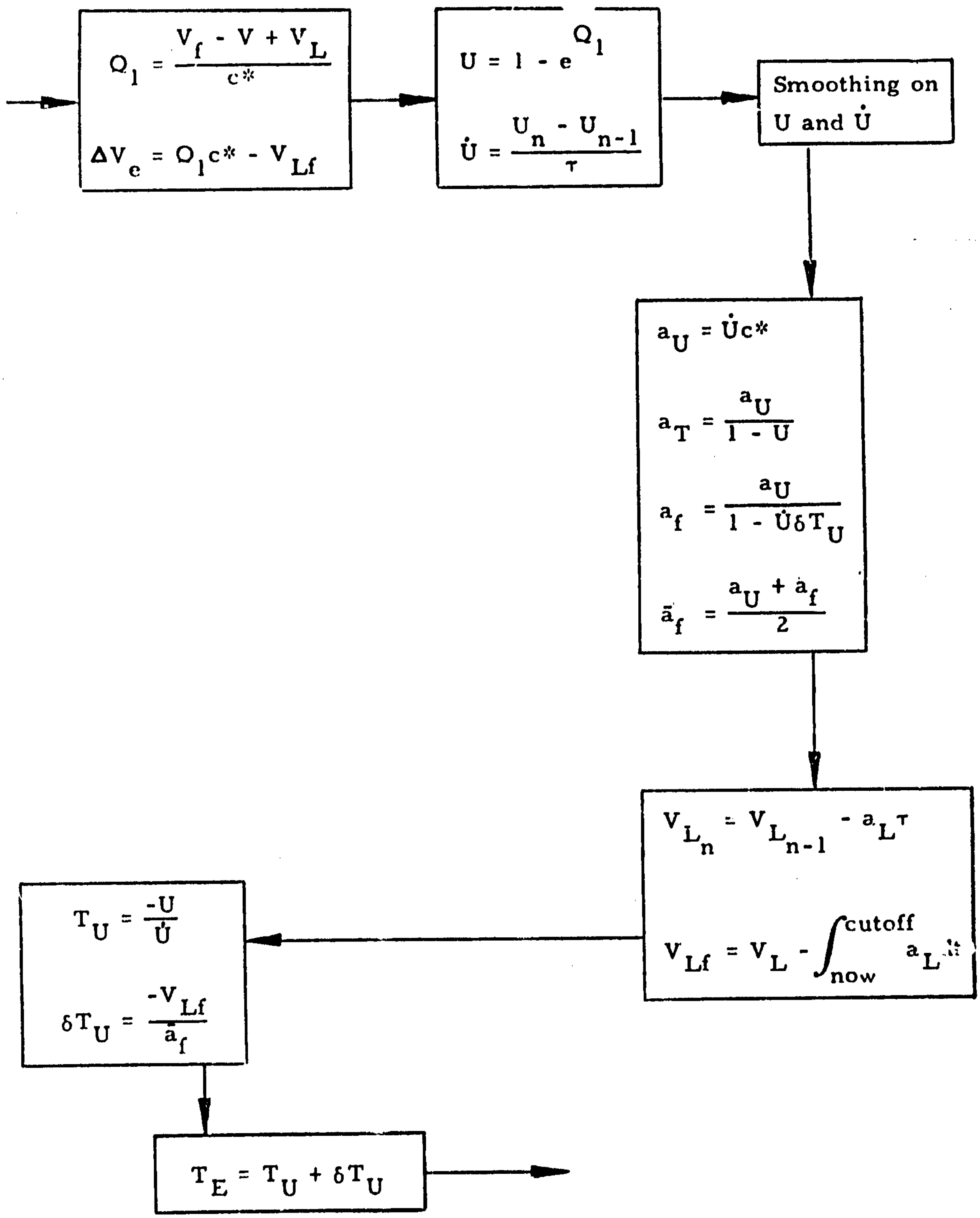
$Q_1 = \frac{V_f - V + V_L}{c^*}$
$\Delta V_e = Q_1 c^* - V_{Lf}$
$U = 1 - e^{Q_1}$
$\dot{U} = \frac{U_n - U_{n-1}}{\tau}$
Smoothing on U and $\dot{U}$
$a_U = \dot{U} c^*$
$a_T = \frac{a_U}{1 - U}$
$a_f = \frac{a_U}{1 - \dot{U}\delta T_U}$
$\bar{a}_f = \frac{a_U + a_f}{2}$
$V_{L_n} = V_{L_{n-1}} - a_L \tau$
$V_{Lf} = V_L - \int_{now}^{cutoff} a_L\,dt$
$T_U = \frac{-U}{\dot{U}}$
$\delta T_U = \frac{-V_{Lf}}{\bar{a}_f}$
$T_E = T_U + \delta T_U$

Figure 2.3

3. EFFECTS OF MISSILE ATTITUDE

Equations relating missile attitude profile and vehicle position and velocity will be developed in this section. These relationships can then be used for control purposes. Pitch plane dynamics will be discussed first as they are the most general. Yaw dynamics will then be a special case of this general solution.

Great mathematical simplicity would result if the development could be made in an inertial cartesian coordinate system. The approach taken in the following derivation will be to solve the problem in such an inertial frame and to add terms to the solution to account for the differences between this model and reality. In order to validly use this approach it is necessary to eliminate the anomalies caused by the following items in the physical environment:

(1) The "true" inertial coordinate system is at the geocentroid and not rotating, while the desired vehicle position is most conveniently specified as a radius from the geocentroid.

(2) The existence of a gravitational field.

Item (1) can be compensated for by use of the following coordinate frame (see Figure 2.1):

Let V = inertial velocity of the vehicle

r = distance from the geocentroid to the vehicle

Then

$$V_p = \frac{\vec{r} \cdot \vec{V}}{|r|}$$

$$\sin \Gamma = \frac{V_p}{V} \qquad (3.1)$$

Since these variables are derived relative to r, they "rotate" with an angular rate $\omega_{Rot} = \frac{V \cos \Gamma}{r}$; but they are truly inertial quantities since they are measured in an inertial coordinate system. (The rotation is "stopped" while the computation is being made.)

Likewise a pitching rate

$$\omega_p^* = \frac{- V \cos \Gamma}{r} \tag{3.2}$$

is necessary to keep thrust attitude (β_p) constant (relative to $\vec{r}$). This pitching rate will be transmitted in addition to any pitching rate defined for other purposes.

The gravitational and coriolis accelerations in these coordinates produce a combined acceleration

$$g_{Ep} = g_r - \frac{[V \cos \Gamma]^2}{r} \tag{3.3}$$

where g_r is the component of gravitational attraction along $\vec{r}$ and may be defined in any desired degree of complexity (the component perpendicular to $\vec{r}$ is of no consequence here, and is negligible in any case). The effect of g_{Ep} can be compensated for by defining a missile attitude

$$\beta_{gp} = \frac{g_{Ep}}{a_T} \tag{3.4}$$

and adding this attitude to the attitude desired for position and velocity control. This specifically assumes that

$$\sin \beta = \beta \tag{3.5}$$

a relationship that will be assumed for si.. ?' city throughout the development. While this approximation is valid for most practical trajectories, less approximate relationships are easily derived (see Appendix H). A pitching rate

$$\omega_{gp} = \frac{d\beta_{gp}}{dt} \tag{3.6}$$

must be transmitted to maintain the correct value of β_{gp}. The differentiation of Equation (3.4) is given in Appendix C.

The equations governing position and velocity in an inertial plane will now be developed. These vehicle attitudes must satisfy the following relationships, using Equation (3.5):

$$V_{pf} = V_{po} + \int_{t_o}^{t_f} a_T \beta_{mp} dt \tag{3.7}$$

$$r_f = r_o + \int_{t_o}^{t_f} V_{po} dt + \int_{t_o}^{t_f}\int_{t_o}^{t} a_T \beta_{mp} dt dt \tag{3.8}$$

A necessary condition to the solution of Equations (3. 7) and (3.8) is that

$$\beta_{mp} = A_o + A_1 f_1 (t_f - t)$$

where A_o and A_1 are arbitrary constants and f_1 is an arbitrary function of $(t_f - t)$, and, of course

$$\omega_{mp} = \frac{d\beta_{mp}}{dt} = A_1 \frac{d}{dt} \left[f_1 (t_f - t) \right]$$

The functional form of $f_1(t_f - t)$ is restricted only in that it must satisfy Equations (3.7) and (3.8); and, in theory, need be neither analytic nor continuous. There are, however, several rather obvious physical reasons for restricting the physical form of $f_1(t_f - t)$. The attitude must be continuous and attitude rates should be small on practical trajectories. These requirements are satisfied by setting

$$f_1(t_f - t) = t_f - t$$

or

$$\beta_{mp} = A_o + A_1(t_f - t)$$

This makes ω_{mp} constant, with the resulting mathematical advantages.

Several developments have been advanced by different writers with various physical assumptions on the problem of maximizing performance; the results of all argue that either β_{mp} or $\sin \beta_{mp}$ or $\tan \beta_{mp}$ should be linear in time. The fact that these "proofs" are not strictly applicable to the real case is often overlooked or not emphasized, but they do indicate that a constant attitude rate solution should be relatively efficient (particularly when g_{Ep} is small relative to a_T). This constant attitude rate solution is generally used and will be developed here. Other types of solutions can be easily developed along the same lines. A development employing two constant attitudes has been used on Ranger/Mariner.

It is possible to choose a constant attitude $\overline{\beta}_p$ which would satisfy Equation (3.7) [without, in general, satisfying Equation (3.8)]. For this attitude Equation (3.7) reduces to

$$V_{pf} = V_{po} + \overline{\beta}_p \int_{t_o}^{t_f} a_T dt = V_{po} + \overline{\beta}_p \Delta V_e \tag{3.9}$$

or

$$\bar{\beta}_p = \frac{V_{pf} - V_{po}}{\Delta V_e} \tag{3.10}$$

Then missile attitude can be expressed as

$$\beta_{mp} = \bar{\beta}_p + \beta_{prf} - \omega_{pr}(t_f - t) \tag{3.11}$$

Equation (3.10) satisfies Equation (3.7) by requiring

$$\int_{t_o}^{t_f} a_T\left[\beta_{prf} - \omega_{pr}(t_f - t)\right]dt = 0 \tag{3.12}$$

The notation ω_{pr} is adopted to signify that it is needed only to provide position (radius) control. The implications and advantages of this formulation of β_{mp} will be discussed following the development. The various β quantities are shown schematically in Figure 3.1.

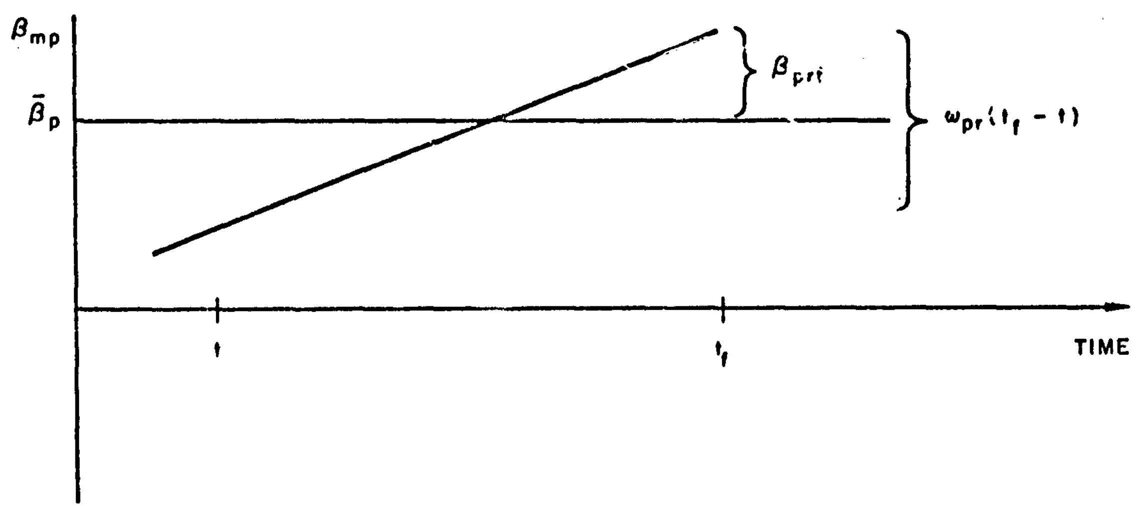

Figure 3.1

Expressions for β_{prf} and ω_{pr} will be found by su .tituting Equation (3.11) into Equation (3.8) and combining with Equation (3.12). This yields (see Appendix B)

$$\omega_{pr} = \frac{r_f - r_o - T_E\left[Q_6(V_{pf} - V_p) + V_p\right]}{\left(Q_6 - \frac{1}{2}\right)\left(T_E\right)^2 c*} \tag{3.13}$$

$$\beta_{prf} = Q_6 T_E \omega_{pr} \tag{3.14}$$

where

$$Q_6 = c*\left(\frac{1}{\Delta V_e} - \frac{1}{a_f T_E}\right) \tag{3.15}$$

The numerator in Equation (3.13) is the altitude "error" that would exist if ω_{pr} were held at zero; the denominator is the appropriate scaling factor between ω_{pr} and this altitude error.

The desired present missile attitude is the sum of the various attitude components which have been generated above, so

$$\beta_o = \beta_{pg} + \overline{\beta}_p + \beta_{prf} - \omega_{pr} T_E = \beta_{pg} + \overline{\beta}_p - (1 - Q_6)\omega_{pr} T_E \tag{3.16}$$

The actual missile attitude β_p which is determined by any suitable method will not in general be equal to β_o (although in "steady state" the difference should be quite small) so that there will be an additional transient pitch turning rate

$$\omega_{pt} = (\beta_o - \beta_p)G \tag{3.17}$$

where G is a gain function. Then the total pitch turning rate is

$$\omega_p = \omega_{pg} + \omega_p^* + \omega_{pr} + \omega_{pt} \tag{3.18}$$

After an initial transient, ω_{pt} will be very small; and ω_{pg} and ω_p^* are always small because of the physics of the problem. The magnitude of ω_{pr} is in effect a measure of the feasibility of satisfying the specified burnout conditions, since if an excessive amount of steering is required it may not be practical or even possible to satisfy these burnout conditions (this is why Equation (3.5) is usually satisfactory). Note that limiting the magnitude of ω_{pr} will modify the burnout altitude requirement <u>without</u> affecting the ability to satisfy the velocity requirement in any way whatsoever. This feature is especially convenient near cutoff, since there is practically no capability to change altitude and the inclusion of a limit on ω_{pr} allows the equations to function in a rational manner without necessitating any other changes.

It should be noted for completeness that the superposition which has been assumed is not strictly valid for non-zero values of V_p. This effect is extremely small for practical thrust levels (burning times), vanishes as T_E goes to zero, and will not be discussed for these reasons.

Figure 3.2 is a collection of equations from this section which are needed for pitch steering. As in Figure 2.3, these equations are not necessarily ideal for mechanization.

Yaw steering may be mechanized in a number of ways. It would be possible, for example, to define a desired orbit plane with displacement y and a yaw velocity V_y perpendicular to this plane. Substitution of appropriate notation in Equations (3.7) through (3.18) is all that is required to make these equations applicable to yaw steering for this case. Equations (3.2), (3.4) and (3.6) are replaced by

$$\beta_{yg} = \omega_{yg} = \omega_y^* = 0 \tag{3.19}$$

a condition that exists for any form of yaw steering.

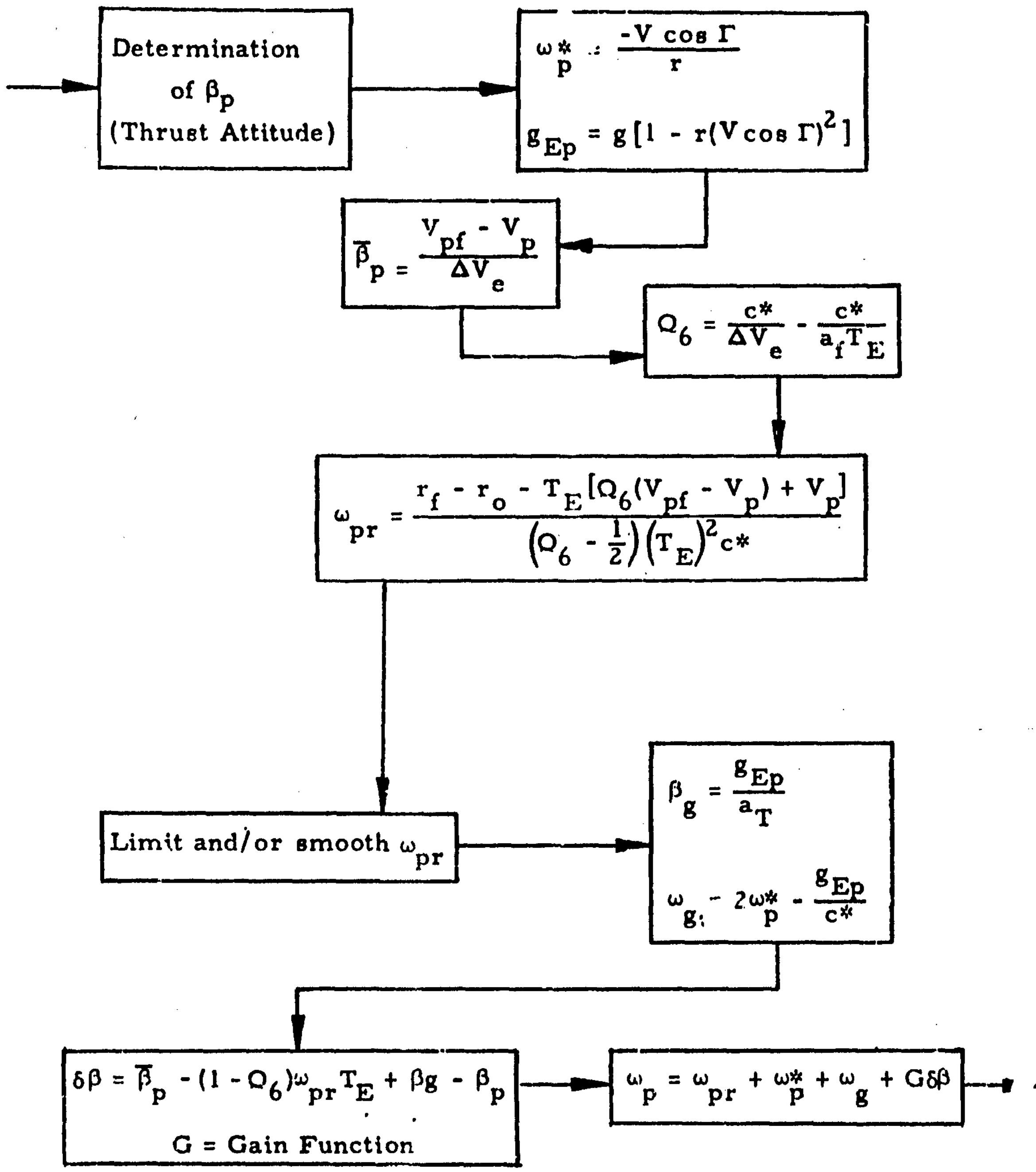
Determination of β_p (Thrust Attitude)
$\omega_p^* = \frac{-V\cos\Gamma}{r}$
$g_{Ep} = g[1 - r(V\cos\Gamma)^2]$
$\bar{\beta}_p = \frac{V_{pf} - V_p}{\Delta V_e}$
$Q_6 = \frac{c^*}{\Delta V_e} - \frac{c^*}{a_f T_E}$
$\omega_{pr} = \frac{r_f - r_o - T_E[Q_6(V_{pf} - V_p) + V_p]}{\left(Q_6 - \frac{1}{2}\right)\left(T_E\right)^2 c^*}$
Limit and/or smooth ω_{pr}
$\beta_g = \frac{g_{Ep}}{a_T}$
$\omega_g = 2\omega_p^* - \frac{g_{Ep}}{c^*}$
$\delta\beta = \bar{\beta}_p - (1 - Q_6)\omega_{pr} T_E + \beta g - \beta_p$
G = Gain Function
$\omega_p = \omega_{pr} + \omega_p^* + \omega_g + G\delta\beta$

Figure 3.2

It may be unnecessary and thus undesi: '.e (from a performance standpoint) to control the orbit plane with the precision available in the above equations. A somewhat simpler technique would be to choose a point in the desired orbit plane (a "pseudo-target") and require that the intersection of the actual orbit plane and the desired orbit plane contain this point. This can be insured by defining a yaw velocity (V_y) perpendicular to the plane containing the missile (r), the target (r_T), and the geocentroid; and then requiring that V_y be zero at cutoff. This may be mechanized by defining

$$V_y = \frac{\vec{r}_T \times \vec{r} \cdot \vec{V}}{r_T r \sin \psi} \tag{3.20}$$

where ψ is the angle between $\vec{r}_T$ and $\vec{r}$. Requiring only that $V_{yf} = 0$ is in effect setting the term corresponding to ω_{pr} equal to zero, and thus replacing Equation (3.16), (3.17) and (3.18) by

$$\omega_y = (\overline{\beta}_y - \beta_y)G \tag{3.21}$$

At cutoff the vehicle will be displaced from the desired burnout plane by approximately

$$\Delta y = \int_{\text{liftoff}}^{\text{cutoff}} V_y dt \tag{3.22}$$

The integrand in Equation (3.22) vanishes at liftoff and cutoff, and, in tl . absence of malfunction, never becomes very large. Since the vehicle is in the desired plane at the pseudo-target, this pseudo-target can for many missions be selected such that the magnitude of Δy is of little or no importance. If the magnitude of Δy is sufficiently small in the presence of normal dispersions (as it usually is) the simplicity and fuel economy resulting from this pseudo-target technique justify its use for these missions.

4. SIMULATIOI .ESULTS

Guidance equations utilizing the preceding explicit solution have been simulated by several people for use in achieving a variety of burnout conditions (both with and without noise) for a variety of missions. In the absence of errors in positions and velocity information, approximate errors of injection into an orbit (if c* is known) are given by the following tabulation:

Parameter	3σ dispersion
Velocity Magnitude	.1 foot per second
Velocity Orientation	$.5(10^{-4})$ radian = .003 degree
Altitude	150 feet

Injection at a relatively large Γ does, for rather obvious reasons, increase the altitude dispersions, although this dispersion still remains small. These dispersions are due to data lag, response lag, smoothing lag (filters are used even in the absence of noise), variability of thrust and c* from assumed constant value due to the use of influence coefficients in the simulation of engine thrust and mass flow, etc. In an actual mission noise and various other hardware type effects (such as cutoff impulse uncertainty) will cause considerably larger injection errors. Some of these errors can be reduced by the use of a vernier.

Lack of knowledge of the exact value of c* will introduce a bias in time-to-go only if smoothing is introduced (see Appendix I). The size of this bias will depend not only on the "error" in c* but also on the "time constant" of the smoothing filter; however, reasonable values of these quantities produce a ve-v small effect on velocity magnitude and negligible effect on altitude and veloc.ity orientation. Errors due to noise will not be discussed except to note that the extreme precision (linearity) of the "time-to-go" solution minimizes the severity of problems arising from this area (absolute linearity would give optimum results).

APPENDIX A
EFFECTS OF ERRORS IN c*

The sensitivities of T_U and a_T to inexact knowledge of value of c* (which is still a constant) will be developed.

For convenience we define

$$Q_1 \equiv \frac{V_f - V + V_L}{c^*} \tag{A.1}$$

so that from Equation (2.13)

$$U = 1 - e^{Q_1}$$

Also from Equations (2.8) and (2.9)

$$a_T = \frac{\dot{U} c^*}{1 - U} \tag{A.2}$$

Therefore:

$$\frac{\partial U}{\partial c^*} = \frac{\partial}{\partial c^*}\left(1 - e^{Q_1}\right) = -e^{Q_1}\frac{\partial Q_1}{\partial c^*} = e^{Q_1}\frac{Q_1}{c^*} = (1 - U)\frac{Q_1}{c^*}$$

Since c* and t are independent

$$\frac{\partial}{\partial c^*}\left(\frac{dU}{dt}\right) = \frac{d}{dt}\left(\frac{\partial U}{\partial c^*}\right) = \frac{d}{dt}\left[(1 - U)\frac{Q_1}{c^*}\right] = \frac{-\dot{U}Q_1}{c^*} + (1 - U)\left(\frac{-a_T}{c^{*2}}\right)$$

and with (A.2)

$$\frac{\partial}{\partial c^*}\,\frac{dU}{dt} = -\frac{\dot{U}}{c^*}\left(1 + Q_1\right) \tag{A.3}$$

Differentiating Equation (A.2)

$$\frac{\partial a_T}{\partial c^*} = \frac{(1 - U)\left[\dot{U} + c^* \frac{\partial}{\partial c^*}\left(\frac{dU}{dt}\right)\right] + \dot{U} c^* \frac{\partial U}{\partial c^*}}{(1 - U)^2} = \frac{-\dot{U}Q_1 + \dot{U}Q_1}{1 - Q} = 0 \qquad (A.4)$$

This is a somewhat surprising, but very gratifying, result.

Differentiating Equation (2.7)

$$T_U = \frac{-U}{\dot{U}} \qquad (2.7)$$

$$\frac{\partial T_U}{\partial c^*} = \frac{-\dot{U}\left(\frac{\partial U}{\partial c^*}\right) + U \frac{\partial}{\partial c^*}\left(\frac{\partial U}{\partial t}\right)}{\dot{U}^2} = \frac{-(1 - U)\frac{Q_1}{c^*} - \frac{U}{c^*}(1 + Q_1)}{\dot{U}} = -\frac{(Q_1 + U)}{\dot{U}c^*}$$

So

$$\frac{\partial T_U}{\partial c^*}\frac{c^*}{T_U} = -\frac{Q_1 + U}{T_U \dot{U}} = 1 + \frac{Q_1}{U} = 1 - \frac{\ln(1 - U)}{-U} \qquad (A.5)$$

From Equation (2.9)

$$\frac{a_U}{a_T} = 1 - U$$

The ratio of mass at present to mass when U = 0 is then

$$\frac{m_o}{m_U} = 1 - U \qquad (A.6)$$

Then

$$\frac{\partial T_U}{\partial c^*}\frac{c^*}{T_U} = 1 - \frac{\ln \dfrac{m_o}{m_U}}{\left(\dfrac{m_o}{m_U}\right) - 1} \tag{A.7}$$

Equation (A. 7) is tabulated below:

$\frac{m_o}{m_U}$	$\frac{\partial T_U}{\partial c^*}\frac{c^*}{T_U}$
1	0
1.5	0.189
2	0.307
3	0.451
4	0.538
5	0.598
10	0.744
∞	1

It is apparent that $\partial T_U / \partial c^*$ is satisfactorily small at all times, and that the percentage error vanishes as T_U goes to zero.

APPENDIX B

INTEGRAL EVALUATION

The notation will be changed in the mathematical development of this appendix to provide greater simplicity and clarity. Final results will be transformed into the original notation. The changes are as follows:

1. The subscript o is used to denote values "now" and unsubscripted values are used to denote values between "now" and final.

2. The time axis is defined so that $t_o = 0$.

From Equation (2.4) the final acceleration is

$$a_f = \frac{a_o}{1 - \frac{a_o}{c^*}(t_f - t_o)} = \frac{1}{\frac{1}{a_o} - \frac{(t_f - t_o)}{c^*}} = \frac{1}{\frac{1}{a_o} - \frac{t_f}{c^*}} \tag{B.1}$$

so that

$$\frac{c^*}{a_f t_f} = \frac{c^*}{a_o t_f} - 1 \tag{B.2}$$

The following relationships will be useful

$$\int_0^t a_T dt = -c^* \ln\left(1 - \frac{a_o}{c^*} t\right) \tag{B.3}$$

$$\int_0^{t_f} a_T dt = \Delta V_e = -c^* \ln\left(1 - \frac{a_o}{c^*} t_f\right) \tag{B.4}$$

Then

$$\int_0^{t_f}\int_0^{t} a_T dt\, dt = \frac{c^{*2}}{a_o}\int_0^{t_f} \ln\left(1 - \frac{a_o}{c^*}t\right)\left(-\frac{a_o}{c^*}dt\right) = c^*\left[\frac{t_f\left(1-\frac{a_o t_f}{c^*}\right)\ln\left(1-\frac{a_o}{c^*}t_f\right)}{\frac{a_o t_f}{c^*}} + t_f\right]$$

which in combination with Equations (B.2) and (B.4); and with the definition

$$Q_5 \equiv 1 - \frac{\Delta V_c}{a_f t_f} \tag{B.5}$$

becomes

$$\int_0^{t_f}\int_0^{t} a_T dt\, dt = c^* t_f\left[\frac{c^*}{a_f t_f}\left(\frac{-\Delta V_c}{c^*}\right) + 1\right] = c^* t_f Q_5 \tag{B.6}$$

From Equation (B.1)

$$a_T = \frac{a_o}{1 - \frac{a_o}{c^*}t}$$

so that

$$\int_0^{t} a_T t\, dt = a_o\int_0^{t} \frac{1}{1 - \frac{a_o}{c^*}t}\, dt = -c^*\left[t + \frac{c^*}{a_o}\ln\left(1 - \frac{a_o}{c^*}t\right)\right]$$

which in combination with Equations (B.2) and (B.5) gives

$$\int_0^{t_f} a_T t\, dt = c^* t_f \left[\frac{\Delta V_e}{c^*} - Q_5\right] \tag{B.7}$$

Also

$$\int_0^{t_f}\int_0^{t} a_T t\, dt\, dt = -c^* \int_0^{t_f}\left[t + \frac{c^*}{a_o}\ell n\left(1 - \frac{a_o}{c^*}t\right)\right]dt$$

$$= -c^*\left[\frac{t_f^{\,2}}{2} - \frac{1}{a_o}\int_0^{t_f}\int_0^{t} a_T\, dt\, dt\right] = c^* t_f^{\,2}\left[-\frac{1}{2} + \frac{c^*}{a_o t_f} Q_5\right] = c^* t_f^{\,2}\left[-\frac{1}{2} + Q_5 + Q_5\left(\frac{c^*}{a_f t_f}\right)\right]$$

Defining

$$Q_6 \equiv \frac{c^* Q_5}{\Delta V_e} \tag{B.8}$$

Then

$$\int_0^{t_f}\int_0^{t} a_T t\, dt\, dt = c^* t_f^{\,2}\left[-\frac{1}{2} + Q_5 + Q_6(1 - Q_5)\right] \tag{B.9}$$

Integrating Equation (3.12)

$$(\beta_{prf} - \omega_{pr} t_f)\int_0^{t_f} a_T dt + \omega_{pr}\int_0^{t_f} a_T t\, dt = 0$$

Combining with Equations (B.4) and (B.7)

$$(\beta_{prf} - \omega_{pr} t_f)\Delta V_e + \omega_{pr} c^* t_f \left(\frac{\Delta V_e}{c^*} - Q_5\right) = 0$$

$$\beta_{prf} = \omega_{pr} t_f \left[1 - \left(1 - \frac{Q_5 c^*}{\Delta V_e}\right)\right] = \omega_{pr} t_f Q_6 \qquad \text{(B.10)}$$

Substituting Equation (B.10) into Equation (3.11)

$$\beta_{mp} = \overline{\beta}_p + \omega_{pr} \left[(Q_6 - 1)t_f + t\right]$$

Substituting this expression in Equation (3.8)

$$r_f = r_o + V_p t_f + \overline{\beta}_p \int_0^{t_f}\int_0^t a_T \, dt\, dt + \omega_{pr}(Q_6 - 1)t_f \int_0^{t_f}\int_0^t a_T \, dt\, dt + \omega_{pr} \int_0^{t_f}\int_0^t t a_T \, dt\, dt$$

Substituting Equations (B.6) and (B.9) and solving for ω_{pr}

$$\omega_{pr} = \frac{r_f - r_o - V_p t_f - \overline{\beta}_p c^* t_f Q_5}{(Q_6 - 1)t_f^2 c^* Q_5 + c^* t_f^2 \left[-\frac{1}{2} + Q_5 + Q_6 - Q_5 Q_6\right]}$$

or

$$\omega_{pr} = \frac{r_f - r_o - t_f(V_p + \overline{\beta}_p c^* Q_5)}{c^* t_f^2 \left(Q_6 - \frac{1}{2}\right)} = \frac{r_f - r_o - t_f\left[Q_6(V_{pf} - V_p) + V_p\right]}{c^* t_f^2 \left(Q_6 - \frac{1}{2}\right)} \qquad \text{(B.11)}$$

The numerical values of t_f and T_E are identical, and making this substitution in Equations (B.11) and (B.10) produces Equations (3.13) and (3.14) respectively. Also

$$Q_6 = \frac{c^*}{\Delta V_e} - \frac{c^*}{a_f T_E} \tag{B.12}$$

APPENDIX C

EVALUATION OF ω_{pg}

The approximation $g = g_r = K_1/r^2$ is valid for reasonably small variations in r and will be assumed valid for purposes of differentiating β_g, although this restriction is not necessarily made in the definition of β_g. With this assumption Equation (3.3) can be written as

$$g_{Ep} = \frac{K_1}{r^2}\,[1 - K_2 r V^2 \cos^2 \Gamma] = \frac{K_1}{r^2}\left[1 - K_2 r\left(V^2 - V_p^2\right)\right] \qquad \text{(C.1)}$$

so that from Equation (3.4)

$$\beta_g = \frac{K_1}{r^2 a_T}\left[1 - K_2 r V^2 + K_2 r V_p^2\right] \qquad \text{(C.2)}$$

Differentiating Equation (C.2)

$$\frac{d\beta_g}{dt} = \frac{K_1}{r^2 a_T}\left[-K_2 V^2 V_p - 2K_2 r V \frac{dV}{dt} + K_2 V_p^3 + 2K_2 r V_p \frac{dV_p}{dt}\right] -$$

$$-\beta_g \left[\frac{2 r a_T V_p + r^2 \dfrac{da_T}{dt}}{a_T r^2}\right]$$

$$= -\frac{gK_2}{a_T}\left[V^2 V_p + 2rV\frac{dV}{dt} - V_p^3 - 2rV_p\frac{dV_p}{dt}\right] - \beta_g\left[\frac{2V_p}{r} + \frac{1}{a_T}\frac{da_T}{dt}\right] \qquad \text{(C.3)}$$

From Equation (2.3)

$$\frac{da_T}{dt} = \frac{a_T^2}{c*}$$

Then

$$\frac{d\beta_g}{dt} = \frac{-gK_2}{a_T}\left[V_p(V^2 - V_p^2) + 2r\left(\frac{VdV}{dt} - V_p\frac{dV_p}{dt}\right)\right] - \beta_g\left[\frac{2V_p}{r} + \frac{a_T}{c*}\right]$$

Neglecting small terms

$$\frac{d\beta_g}{dt} = -\left[gK_2 2rV + \frac{g_{Ep}}{c*}\right] = -\left[\frac{2V}{r} + \frac{g_{Ep}}{c*}\right]$$

or with Equation (3.2)

$$\omega_{pg} = \frac{d\beta_g}{dt} \approx 2\omega_p^* - \frac{g_{Ep}}{c*} \qquad (C.4)$$

It may be easily shown that

$$\frac{d\omega_g}{dt} \approx \frac{2a_T}{r}\left(\frac{V}{c*} - 1\right) \qquad (C.5)$$

The fact that $d\omega_g/dt$ is extremely small is fortunate for obvious reasons.

APPENDI. J

EVALUATION OF V_{Lf}

In order to mechanize this solution for guidance, it is necessary to mechanize a solution for V_{Lf}. A mechanization which has proven practical will be outlined below.

As before

$$V_{Lf} = V_L - \int_t^{t_f} a_L dt \tag{2.18}$$

where

$$a_L = g \sin \Gamma + a_T \left[1 - \cos (\beta - \Gamma) \right] - \frac{F_A}{m} \cos \Phi \tag{2.10}$$

The integral of a_L w... be approximated term by term. Defining

$$\begin{aligned} a_{Lg} &\equiv g \sin \Gamma \\ a_{L\lambda} &\equiv a_T \left[1 - \cos (\beta - \Gamma) \right] \\ a_{LA} &\equiv \frac{F_A}{m} \cos \Phi \end{aligned} \tag{D.1}$$

and assuming

$$\Gamma = \Gamma_f + \Gamma' (t - t_f) \tag{D.2}$$

$$\beta = \beta_f + \beta' (t - t_f)$$

Then

$$\int_t^{t_f} a_{Lg} dt = \int_t^{t_f} g \sin \Gamma dt \approx g \int_t^{t_f} [\Gamma_f + \Gamma' (t - t_f)] dt$$

$$= g\left(t_f - t\right)\left[\Gamma_f + \Gamma' \frac{(t - t_f)}{2}\right] = g\left(t_f - t\right)\left(\frac{\Gamma + \Gamma_f}{2}\right)$$

or

$$\int_t^{t_f} a_{Lg} dt = gT_E\left(\frac{\Gamma + \Gamma_f}{2}\right) \qquad \text{(D. 3)}$$

Defining

$$\lambda \equiv 1 - \cos (\beta - \Gamma) \approx \frac{(\beta - \Gamma)^2}{2} \qquad \text{(D. 4)}$$

Then with Equation (D. 2)

$$\int_t^{t_f} a_{L\lambda} dt = \int_t^{t_f} a_T \lambda dt = \frac{1}{2}\int_t^{t_f} a_T \left[(\beta_f - \Gamma_f)^2 + 2(\beta_f - \Gamma_f)(\beta' - \Gamma')(t - t_f) + (\beta' - \Gamma')^2 (t - t_f)^2\right] dt$$

$$2\int_t^{t_f} a_T \lambda dt = (\beta_f - \Gamma_f)^2 \int_t^{t_f} a_T dt - 2(\beta_f - \Gamma_f)(\beta' - \Gamma') \int_t^{t_f} a_T (t_f - t) dt + (\beta' - \Gamma')^2 \int_t^{t_f} a_T (t_f - t)^2 dt$$

Using conventions of Appendix B

$$2\int_{t}^{t_f} a_T\lambda dt = \left[(\beta_f - \Gamma_f)^2 - 2(\beta_f - \Gamma_f)(\beta' - \Gamma')t_f + (\beta' - \Gamma')^2 t_f^2\right]\int_{o}^{t_f} a_T dt +$$

$$+\left[2(\beta_f - \Gamma_f)(\beta' - \Gamma') - (\beta' - \Gamma')^2 2t_f\right]\int_{o}^{t_f} a_T t dt +$$

$$+ (\beta' - \Gamma')^2\int_{o}^{t_f} a_T t^2 dt$$

From Eqs. (B. 1), (B. 4), (B. 7), (B. 8), (D. 2) and (D. 4)

$$\int_{t}^{t_f} a_T\lambda dt = \lambda\Delta V_e + (\beta - \Gamma)(\beta' - \Gamma')(t_f - t)\Delta V_e(1 - Q_6) +$$

$$+ \frac{(\beta' - \Gamma')^2}{2}\int_{o}^{t_f} \frac{a_o t^2 dt}{1 - \frac{a_o}{c^*}t} \qquad \text{(D. 5)}$$

Evaluating the integral using the notation of Appendix B

$$\int_{o}^{t_f} \frac{a_o t^2 dt}{1 - \frac{a_o}{c^*}t} = -c^*\left(\frac{c^*}{a_o}\right)^2\left\{\frac{1}{2}\left[\left(1 - \frac{a_o}{c^*}t_f\right)^2 - 1\right] - 2\left(-\frac{a_o t_f}{c^*}\right) + \ln\left(1 - \frac{a_o}{c^*}t_f\right)\right\}$$

$$= -c^*\left(\frac{c^*}{a_o}\right)^2\left[\frac{a_o t_f}{c^*} + \frac{1}{2}\left(\frac{a_o t_f}{c^*}\right)^2 + \ln\left(1 - \frac{a_o}{c^*}t_f\right)\right]$$

With Eq. (B. 4)

$$\int_0^{t_f} \frac{a_o t^2}{1 - \frac{a_o}{c^*} t}\, dt = c^* t_f^2 \left[\frac{\Delta V_e}{c^*} \left(\frac{c^*}{a_o t_f} \right)^2 - \left(\frac{c^*}{a_o t_f} \right) - \frac{1}{2} \right]$$

From Eq. (B. 2), (B. 5) and (B. 7)

$$\frac{c^*}{a_o t_f} = 1 + \frac{c^*}{a_f t_f} = 1 + \frac{c^*(1 - Q_5)}{\Delta V_e} = 1 - Q_6 + \frac{c^*}{\Delta V_e}$$

$$\left(\frac{c^*}{a_o t_f} \right)^2 = (1 - Q_6)^2 + 2(1 - Q_6) \frac{c^*}{\Delta V_e} + \left(\frac{c^*}{\Delta V_e} \right)^2$$

Then

$$\int_0^{t_f} \frac{a_o t^2}{1 - \frac{a_o}{c^*} t}\, dt = c^* t_f^2 \left[\frac{\Delta V_e}{c^*} (1 - Q_6)^2 + 2(1 - Q_6) + \frac{c^*}{\Delta V_e} - 1 + Q_6 - \frac{c^*}{\Delta V_e} - \frac{1}{2} \right] =$$

$$= \Delta V_e t_f^2 \left[(1 - Q_6)^2 + \frac{\frac{1}{2} - Q_6}{\frac{\Delta V_e}{c^*}} \right] \tag{D. 6}$$

Defining

$$Q_7 \equiv (\beta' - \Gamma')(t_f - t) = (\beta_f - \Gamma_f) - (\beta - \Gamma) \tag{D. 7}$$

Then with Eq. (D.6) and (D.7) Eq. (D.5) becomes

$$\int_t^{t_f} a_T \lambda dt = \Delta V_e \left\{ \lambda + (\beta - \Gamma) Q_7 (1 - Q_6) + \frac{Q_7^2}{2} \left[(1 - Q_6)^2 + \frac{\frac{1}{2} - Q_6}{\frac{\Delta V_e}{c^*}} \right] \right\}$$

Then

$$\int_t^{t_f} a_{L\lambda} dt = \bar{\lambda} \Delta V_e \tag{D.8}$$

Where

$$\bar{\lambda} = \lambda + (\beta - \Gamma) Q_7 (1 - Q_6) + \frac{Q_7^2}{2} \left[(1 - Q_6)^2 + \frac{\frac{1}{2} - Q_6}{\frac{\Delta V_e}{c^*}} \right]$$

Experience shows that the following approximation gives very accurate results

$$\bar{\lambda} = \lambda + \frac{(\beta - \Gamma)}{2} Q_7 + \frac{Q_7^2}{6} \tag{D.9}$$

where Q_7 is given in Equation (D.7).

Although aerodynamic forces could be handled in a similar manner, for trajectories normally considered it is very satisfactory to set

$$F_A = a_{LA} = 0 \qquad \text{(D. 10)}$$

Then Equation (2. 18) can be written as

$$V_{Lf} = V_L - gT_E\left(\frac{\Gamma + \Gamma_f}{2}\right) - \Delta V_e\overline{\lambda} \qquad \text{(D. 11)}$$

where $\overline{\lambda}$ is defined in Equation (D. 9).

APPENDIX E

EVALUATION OF $\bar{a}_f$

From Equation (2.16)

$$\frac{\bar{a}_f}{a_U} = \frac{\ell n(1 - \dot{U}\delta T_U)}{-\dot{U}\delta T_U} = 1 + \frac{\dot{U}\delta T_U}{2} + \frac{(\dot{U}\delta T_U)^2}{3} + \frac{(\dot{U}\delta T_U)^3}{4} + \ldots \qquad (E.1)$$

From Equation (2.9)

$$\frac{a_f}{a_U} = \frac{1}{1 - \dot{U}\delta T_U} = 1 + \dot{U}\delta T_U + (\dot{U}\delta T_U)^2 + (\dot{U}\delta T_U)^3 + \ldots$$

so that

$$\frac{a_U + a_f}{2} = a_U\left[1 + \frac{\dot{U}\delta T_U}{2} + \frac{(\dot{U}\delta T_U)^2}{2} + \frac{(\dot{U}\delta T_U)^3}{2} + \ldots\right]$$

Then

$$\frac{a_U + a_f}{2} - \bar{a}_f = a_U\left[\frac{(\dot{U}\delta T_U)^2}{6} + \frac{(\dot{U}\delta T_U)^3}{4} + \ldots\right] \qquad (E.2)$$

It is obvious from Equation (E.2) that the approximation

$$\bar{a}_f = \frac{a_U + a_f}{2} \qquad (E.3)$$

is very good. It can be easily shown that (for $c^* \approx 10,000$ ft/sec) the error in injection velocity incurred by using Equation (r , is approximately

$$\delta V = \left(\frac{V_{Lf}}{800}\right)^3$$

where δV and V_{Lf} are in units of ft/sec. Usually $V_{Lf} << 800$ ft/sec.

APPENDIX F

EFFECTS OF COMPUTATION CYCLE LENGTH AND AUTOPILOT RESPONSE

The injection errors in velocity magnitude and yaw velocity are very small (less than 1 ft/sec.) for all computing cycle lengths less than 10 sec. Pitch velocity errors as a function of major computation cycle time have also been determined from simulation and are shown in Figure F-1. Position errors are correspondingly small. These error numbers do not include any errors except those arising from the guidance equations themselves. The hardware contribution to the injection errors will be relatively independent of computation cycle but will be somewhat larger for the longer computing intervals. It is assumed in all cases that the determination of position and velocity information will be done frequently enough to prevent computation error build-up. For the longer computation cycle times, this may require the up-dating of position and velocity information in sub-cycles. It should be noted that the errors shown in Figure F-1 are to a certain extent due to lack of sophistication in the mechanization of ω_p^* and ω_{pg} (these quantities are not constants during the computation cycle). These errors could be greatly reduced by the use of offsets or a more sophisticated mechanization.

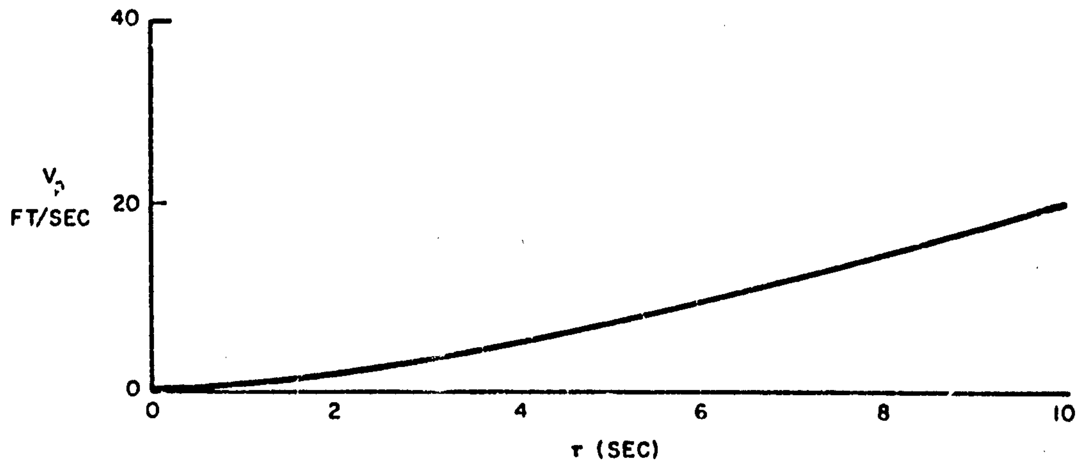

Figure F-1

The effects of lack of knowledge of autopilot responses have been approximately simulated on the IBM 7090. The simulation which was used has a unity autopilot with a delay of any desired magnitude. The guidance equations assume a delay of 2 seconds. Simulations were run with actual delays on the dynamics varying from 0 to 6 sec. which is, of course, far outside any possible lack of knowledge of response characteristics.

When errors in pitch velocity occur due to very long computing cycles or because autopilot response is grossly misjudged, pitch steering commands can assume large or even limited values near vehicle thrust termination. These turning commands as well as the other steering commands used during the flight are not oscillatory and in this sense are not unstable. The existence of this stability has been not only proven by simulation, but may be deduced as follows from past experience and knowledge of system behavior.

In a radio guidance system, vehicle thrust attitude must be deduced from radar position and velocity information. The radar noise normally produces estimates of attitude that are quite inaccurate, but these attitude estimates are then smoothed in a digital filter with a sufficiently long time constant to permit a reasonably accurate attitude to be obtained. This technique works very well (as has been demonstrated on Project Mercury) even when considerable noise is present in the radar data. When an inertial system is used, there may be corresponding "noise" in a position and velocity measured by the IMU. These errors result from rotational dynamics and from the fact that the IMU is not located at the center of gravity of the vehicle as well as quantization of measurements. Although these errors are considerably smaller than the corresponding noise errors in a radio guidance system, the technique for handling them can be the same; i.e., raw attitudes are obtained from position and velocity information and smoothed appropriately. Since the "noise" errors on an inertial system are much smaller than those for a radio guidance system, the problems arising from this area are correspondingly smaller. Since the only data required by the guidance equations are position and velocity information, the above technique will insure stability.

In some guidance techniques not based on an explicit powered flight solution it is not possible or practical to smooth thrust attitude, and this can lead to instability under certain circumstances when these less sophisticated techniques are used.

APPENDIX G

THRUST ATTITUDE DETERMINATION

The following technique has proven very successful as a means of thrust attitude determination. Yaw is a special case of the general pitch relationships.

Pitch attitude is given by

$$\beta_p = \frac{V_{pn} - V_{pn-1}}{\tau a_T} + \beta_{gp}$$

The total turning commands (relative to the local horizontal) which have been generated from the start of guidance are computed as

$$Q_{2n} = Q_{2n-1} + \tau(\omega_p - \omega_p^*) \qquad (G.1)$$

Then the initial attitude (attitude at the time guidance was started) is

$$\beta_p^* = \frac{V_{pn} - V_{pn-1}}{\tau a_T} + \beta_{gp} - Q_{2n} \qquad (G.2)$$

The magnitude of β_p^* should be constant (and simulation has shown it to be remarkably so) and can therefore be readily smoothed.

$$\bar{\beta}_p^* = \beta_p^* \text{ smoothed} \qquad (G.3)$$

Then smoothed missile attitude is

$$\beta_p = \bar{\beta}_p^* + Q_2 \qquad (G.4)$$

APPENDIX H

LESS APPROXIMATE RELATIONSHIPS

More general equations which partially remove the approximation of Equation (3.5) will be given without discussion.

Equation (3.4) becomes

$$\sin \beta_{gp} = \frac{g_{Ep}}{a_T} \tag{H.1}$$

Equation (3.10) becomes

$$\sin \overline{\beta}_p = \frac{V_{pf} - V_p}{\Delta V_e} \tag{H.2}$$

Equation (3.16) becomes

$$\sin \beta_o = \sin \overline{\beta}_p + \sin \beta_{gp} - (1 - Q_6)\omega_{pr} T_E \tag{H.3}$$

Equation (3.17) becomes

$$\omega_{pt} = [\sin \beta_o - \sin \beta_p] G \tag{H.4}$$

Equation (3.18) becomes

$$\omega_p = \omega_p' + \frac{[\omega_{pg} + \omega_{pr} + \omega_{pt}]}{\cos \beta_p} \tag{H.5}$$

Equation (G.2) becomes

$$\overline{\beta}^*_p = \sin^{-1}\left[\frac{V_{pn} - V_{pn-1}}{a_T \tau} + \beta_{gp}\right] - \Omega_{2n} \tag{H.6}$$

NOTATION

The notation has been subdivided into symbols and subscripts. Some of the symbols appear only in combination with subscripts.

SYMBOLS

a - Thrust acceleration

$\bar{a}_f$ - Average thrust acceleration over the time interval δT_U

c^* - Effective exhaust velocity

F_A - Aerodynamic force

F_T - Thrust

g - The total force exerted on the vehicle by gravitational fields divided by the mass of the vehicle

m - Mass of the vehicle including unexpended propellants

r - Distance from the geocentroid to the vehicle

t - Time

T_E - Time to go until the desired velocity is attained

T_U - Time to go until $U = 0$

$\delta T_U = T_E - T_U$

V - Inertial velocity of the vehicle

U - Fundamental dimensionless variable - defined in Equation (2.5), computed from Equation (2.13)

β - Missile attitude

β_m - Component of thrust attitude necessary for position and velocity control in an inertial cartesian coordinate frame

Γ - Flight path angle (positive up from local horizontal)

τ - Length of computational cycle

SYMBOLS (continued)

ω - Vehicle attitude turning rate

ω_p^* - See Equation (3.2)

SUBSCRIPTS

f - Final (when velocity equals desired velocity)

g - Associated with gravity

p - Pitch

r - Associated with geocentric radius

t - Transient

y - Yaw

E - Effective

L - Associated with acceleration "loss"

o - Present value

T - Thrust

U - Related to U

α - Associated with angle of attack